Scaling Exponents for Driven Two-Dimensional Surface Growth

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We present results of numerical simulations to estimate scaling exponents associated with driven surface growth in two spatial dimensions. We have simulated the restricted solid-on-solid growth model and used the time- and system-size-dependent interface width and the equal-time height correlation function to determine the exponents. We also discuss the influence of various functional fitting ansatzes to the correlation function. Our best estimates agree with the results of Forrest and Tang obtained for a different growth model.

Kinetic roughening of interfaces is a ubiquitous phenomenon which takes place under a variety of nonequilibrium conditions.[1] One of the most commonly used examples of a class of systems undergoing this process is driven surface growth far from equilibrium, where the surface current is nonconserved and diffusion rates are very slow compared to the driving force.[2] In this case the relevant mapping of the problem in the continuum limit is given by the Kardar–Parisi–Zhang (KPZ) equation[3]

$$\frac{\partial h}{\partial t} = v \frac{\partial^2 h}{\partial \mathbf{r}^2} + \frac{\lambda}{2} \left(\frac{\partial h}{\partial \mathbf{r}} \right)^2 + \eta \tag{1}$$

where v and λ are constants, and η is a random noise term with

$$\langle \eta(\mathbf{r}, t) \, \eta(\mathbf{r}', t') \rangle = 2D \, \delta^2(\mathbf{r} - \mathbf{r}') \, \delta(t - t')$$

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The height variable $h(\mathbf{r}, t)$ describing the surface is a function of time and two-dimensional vector $\mathbf{r}$, and the total dimensionality is denoted by $d = 2 + 1$.

Due to the scaling relation for the interface width[4]

$$w(L, t) \sim L^{\chi} f(t L^{-z}) \tag{2}$$

where L is the system size, and the scaling relation $z + \chi = 2$, there is only one independent scaling exponent for the problem. It is usually determined either from $w(t) \sim t^{\beta}$, where $\beta = \chi/z$, or from the steady-state limit $w(L) \sim L^{\chi}$.

The scaling exponents are exactly known only in $d = 1 + 1$ dimensions, where $\beta = 1/3$.[3] Much analytic and numerical work has been carried out in order to establish general, dimension-dependent values for the exponents.[1] Monte Carlo simulations of discrete growth models have proven useful for this purpose. For example, extensive work[5] on the hypercubic stacking model has given $\beta(3) = 0.240(1)$ and $\beta(4) = 0.180(5)$. Ala-Nissila et al.[6] simulated the restricted solid-on-solid growth (GRSOS) model up to $d = 7 + 1$. By concentrating on $d \geqslant 3 + 1$, they obtained $\beta(4) = 0.180(2)$, in excellent agreement with ref. 5, but not with the conjecture $\beta(d) = 1/(d + 1)$,[7] and showed that there is no upper critical dimension up to $d = 7 + 1$. Their high-accuracy data for $d = 3 + 1$ were based on a novel fitting ansatz for the equal-time correlation function

$$G(r, t) \equiv \langle [h(r' + r, t) - h(r', t)]^2 \rangle_{r'} \tag{3}$$

averaged over r', as

$$\hat{G}_1(r, t) = a_1(t) \{ \tanh[b_1(t) r^{\gamma_1(t)}] \}^{x_1} \tag{4}$$

where $a_1(t)$, $b_1(t)$, and $\gamma_1(t) \equiv 2\hat{\chi}_1(t)/x_1$ are fitting parameters, and x_1 is fixed. This functional form gives, after fixing x_1, an estimate of β through $a_1(t) \sim t^{2\beta}$, and also for $\chi \approx \hat{\chi}_1$ and z.[6]

In this note, our purpose is to present new simulation data for the GRSOS model in two spatial dimensions and estimate the corresponding scaling exponents. This case is particularly interesting for its potential for experimental realizations and also from a theoretical point of view. In contrast to other recent work claiming $\beta = 0.25$,[8] our best estimate $\beta(3) = 0.240(2)$ is in excellent agreement with the hypercubic stacking model,[5] although finite-size effects seem to be somewhat pronounced. We also discuss the influence of different forms of fitting functions, in addition to Eq. (4), to exponents extracted from the time-dependent correlation function.

Table I. Results of Least-Squares Fitting to Time-
and System-Size-Dependent Width $w^2(L, t)^a$

L	$\beta(3)$	Number of runs
100	0.226(1)	3000
200	0.231(2)	2750
300	0.232(2)	400
1000	0.236(2)	25
2000	0.239(3)	10

[a] Error bars are purely statistical.

First, we used the time-dependent width $w^2(t)$ for several system sizes to determine β. The results of least-squares fitting are presented in Table I. In Fig. 1 we show these values plotted against $1/L$. The result for the largest system size studied, $L = 2000$, already comes very close to 0.240 as obtained in ref. 5, although statistical errors increase considerably for the largest systems. We note that analysis of the data in the form of $\log[w^2(2t) - w^2(t)]$ vs. $\log(t)$ as in ref. 5 failed to produce any consistent results.

To corroborate these findings, we next calculated the saturated width $w^2(L) \sim L^{2\chi}$ for $L = 125$, 250, and 500. These data give $w^2 = 5.15(2)$, 8.67(2), and 15.1(8), where the errors have been estimated from fluctuations between consecutive runs. From a least squares fit we obtain $\chi(3) = 0.387(2)$, which gives $\beta(3) = 0.240(2)$, in complete accordance with the time-dependent width.
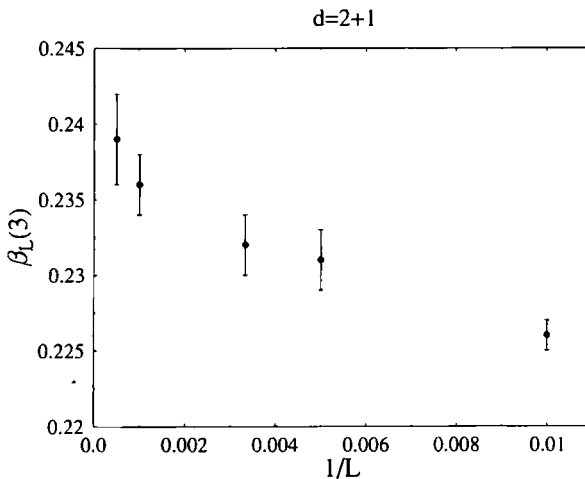
$$d = 2 + 1$$

Fig. 1. Estimates of $\beta(3)$ as obtained from least-squares fits to $w^2(t)$ (see Table I for details).

As previously shown,[6] the fitting ansatz (4) can be used to obtain accurate estimates of β even for relatively small systems. In ref. 6, accurate results for β were obtained by using Eq. (4). It was also shown that the values of $\hat{\chi}_1$ obtained were somewhat smaller than those corresponding to β [as extracted from $a_1(t)$], except in $d = 1 + 1$ dimensions. In the previous work, x_1 was fixed to be $x_1 = 1$ ($d = 1 + 1$) or $1/2$ ($d \geqslant 3 + 1$). In this work, we let x_1 vary and also extend the original fitting ansatz to include the following new fitting functions:

$$\hat{G}_2(r, t) = a_2(t)\{1 - \exp[-b_2(t)r^{\gamma_2(t)}]\}^{x_2} \tag{5}$$

and

$$\hat{G}_3(r, t) = a_3(t)\left\{-\frac{\pi}{4} + \arctan[\exp(-b_3(t)r^{\gamma_3(t)})]\right\}^{x_3} \tag{6}$$

where $a_2(t)$, $a_3(t)$, $b_2(t)$, $b_3(t)$, $\gamma_2(t) \equiv 2\hat{\chi}_2(t)/x_2$, and $\gamma_3(t) \equiv 2\hat{\chi}_3(t)/x_3$ are new fitting parameters. To perform the fitting, we calculated 3000 averages of the correlation function (3) for $L = 100$ and 540 averages for $L = 200$ and 500.

To test the quality of the fitting functions, we first fixed $\hat{\chi}_i = 0.387$ for each function and allowed x_i to vary. Data for $L = 200$ were used. For each function, then x_i was fixed corresponding to an average value obtained between 248 and 600 Monte Carlo time steps per site (MCS/s), where typical variations of x_i are less than about 10%. This gives $x_1 = 0.7389$, $x_2 = 0.4781$, and $x_3 = 0.6484$, which values were in turn used to obtain $\hat{\chi}_1 = 0.387(8)$, $\hat{\chi}_2 = 0.387(5)$, and $\hat{\chi}_3 = 0.386(8)$, respectively, as average values over 248–800 MCS/s. Next, we calculated estimates for the exponent z, averaging results for $c = 0.9$ and 0.95 over 248–800 MCS/s (as explained in ref. 6), and obtained $z = 1.60(2)$, 1.62(2), and 1.61(2) for fitting functions (4), (5), and (6), respectively. These values obey the scaling relation $\hat{\chi} + z$, giving 1.99(3), 2.01(3), and 2.00(3), respectively. This demonstrates that the quality of each fitting function is very good.

Next, we fixed x_2 as above and calculated $a_2(t)$ for the fitting function of Eq. (5), which gave the smallest variation for $\hat{\chi}$. Results for the other functions are consistent. For $L = 100$ and 200, both the original correlation function and the fitting data are very smooth, and least squares fitting to $a_2(t)$ gives $\beta = 0.242(2)$ and 0.240(2), respectively. For $L = 500$, the data are not as good, but fitting to a relatively straight region of the log–log curve gives 0.238(1) (the error bars are purely statistical). These values are in excellent agreement with data obtained from the width for the largest systems in Table I.

As a final check, we calculated the scaling function of Eq. (2) as shown in Figs. 2a and 2b. For the range of system sizes studied in Table I, $\beta = 0.24$ in Fig. 2a gave clearly better scaling than $\beta = 0.25$ of Fig. 2b. In the inset of each figure, we also show the scaling functions for $L = 1000$ and 2000 only. For these two largest system sizes, the accuracy of the data do not allow us to distinguish between the two values of the exponent.

To summarize, we have presented results of rather extensive simulations of the GRSOS model for system sizes up to $L = 2000$. Our best estimate $\beta(3) = 0.240(2)$ comes out from various independent ways of determining the scaling exponents, and is in excellent agreement with ref. 5 for the range
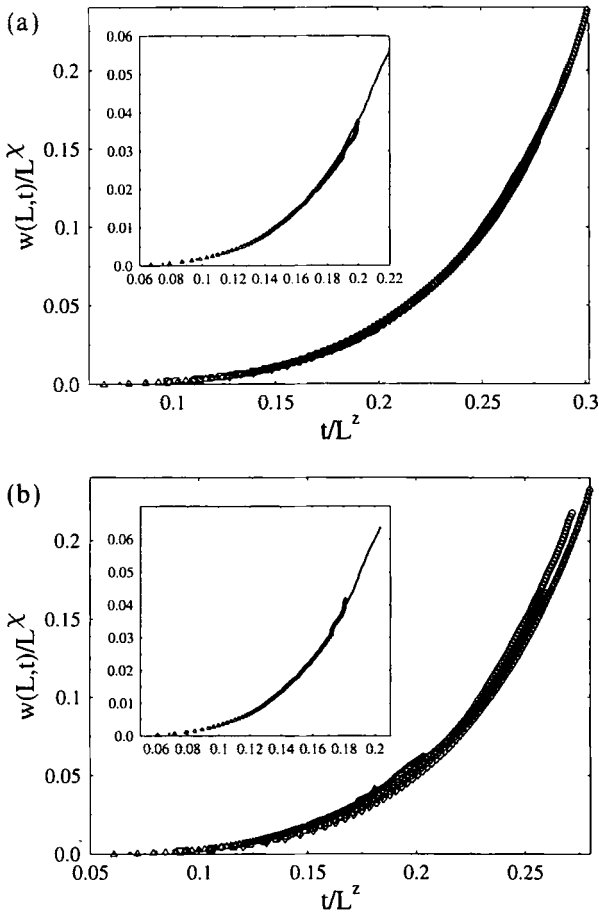

Fig. 2.   (a) Scaling of $w(L, t)$ for $L = 100$, 200, 300, 1000, and 2000, with $\beta = 0.24$, and (b) the same data for $\beta = 0.25$. Insets show scaling between $L = 1000$ and 2000 only.

of system sizes considered in the present work. Unfortunately, a straight-forward extension of this work to larger systems becomes prohibitive, since several hundred hours of computer time in RISC workstations was required here already.

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NOTE ADDED IN PROOF

A single run for a $L = 4000$ system gives results fully consistent with $L = 2000$, with β increasing at later times accompanied by very large fluctuations in $w(t)$.

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